INTRODUCTION TO LINEAR ALGEBRA PART 2

BY SIYATHOKOZA NGEMA

" BE THE CHANGE THAT YOU WISH TO SEE IN THE WORLD"

[PART 2] This document is designed to simplify the concept of matrices for mathematical science students who aim to deepen their understanding of this topic. Matrices are an essential part of linear algebra, typically studied at the second or third-year level. The purpose of this document is to provide mathematical science students with a solid foundation in matrices, ensuring they do not encounter difficulties when advancing to more complex topics in linear algebra. SIYATHOKOZA NGEMA, currently a student at the University of Zululand, has written this document to share his understanding of matrices. "I hope you find this document insightful and beneficial for learning matrices."

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PART 2 (INTRODUCTION TO LINEAR ALGEBRA [MATRICES])

BY SIYATHOKOZA NGEMA

Linear Equations in Matix Form

Any equation or inequality can be written in matrix notation by using what we call matrix multiplication

Example 3x+2y=5

Can be written as

$\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$=5

(1x2) (2x1) (1x1) →dimensions (matrix sizes)

So, if we define A=$\begin{bmatrix} 3 & 2 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \end{bmatrix}$ B= [5] we obtain AX=B

Similarly, the simultaneous equations

$5x_1$+$2x_2$ =10

$4x_1 - 3x_2$=1

Can be written as $\begin{bmatrix} 5 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 10 \\ 1 \end{bmatrix}$

(2x2) (2x1) (2x1)→ $matrix\ sizes$

And if we define A=$\begin{bmatrix} 5 & 2 \\ 4 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **and B=**$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$

we get AX=B

We can use the same idea for any number of equations

Now let show how this idea can be useful given

x+2y=1

x+3y=3

so, we want to solve for x and y using the same idea I have just presented

consider $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ **i.e. AX=B we want to find X**

if the RHS of the equation is premultiplied by the inverse of A , A^{-1},then the LHS must also be premultiplied by A^{-1} to maintain a ' balanced' equation A^{-1}AX= $A^{-1}B$

now we know that the product of a matrix and its inverse is the identity matrix

∴ A^{-1}AX= $A^{-1}B$ →$IX=A^{-1}B$

Now important we must know that multiplying a matrix by the identity does not change the matrix

∴ IX= $A^{-1}B$

→$X=A^{-1}B$

Thus we can conclude that we have found a way of finding the vector of unknowns i.e. we can obtain X.

What we need to do is to premultiply vector B (i.e. the RHS of the original equations by the inverse of the matrix of coefficients from the LHS of the original equations

Now it time to prove that this does indeed work.

NB

1. You could do this graphically or by substitution/Elimination without much trouble but let say there were 10,20… n equations to solve-your existing techniques would indeed be tedious! using matrix algebra we can find a simple technique which can easily be programmed for the computer so that the number of equations become irrelevant so power to matrix algebra and the mathematicians who invented it

Example 2

Solve x+2y=1 …..⊙

x+3y=3……⊙

using substitution or any other technique you know.

Solution: subtract equation 1 from equation 2

$\rightarrow y = 2$

And substituting in 1 for y and the solving for x

If y=2, x+2y=1 gives

X+4=1

$\therefore x = -3$

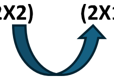
Now using matrix algebra to solve the equations in example 2 above.

A=$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **IS** $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

If X=A^{-1}**B**

X=$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(2X2) (2X1)



Same 2=2 $\therefore$ *product defined then the result will be the outer dimensions* **(2x 1)**

So X=$\begin{bmatrix} (3 \times 1 + & -2 \times 3) \\ (-1 \times 1 + & 1 \times 3\,) \end{bmatrix}$=$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix}$=$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $\therefore$ **x= −3 and y=2**

So to check the solution substitute in each of the original equations

Exercise 1

1. **Write each of the following set of simultaneous equations in matrix form.(define all matrices and vectors)**

   (a) $-4x + 3y = 8$

   3x−2y=12

   (b) $-x - 2y + 3z = -4$

   y$-z = 6$

   2x$-y - 2z = 2$

2.Now use matrix algebra to find the solution to each set of equations [hint: use the inverses you found in example 2]

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY ROW REDUCTION [GAUSS-JORDAN TECHNIQUE]

Before moving on to find the inverse of a matrix we need to introduce the Gauss-Jordan or row reduction technique for solving a system of linear equation as we will use this technique to find the inverse of matrices

We'll demonstrate the technique with a small system of two equations.

2x−y= 3……⊙

x + y= 9……⊙

$\Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$=$\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ **In matrix notation**

$\Rightarrow \begin{bmatrix} 2 & -1 & \vdots & 3 \\ 1 & 1 & \vdots & 9 \end{bmatrix}$ **Augmented matrix**

$\Rightarrow \begin{bmatrix} 1 & 1 & \vdots & 9 \\ 2 & -1 & \vdots & 3 \end{bmatrix}$ **R1**$\leftrightarrow R2$

$\Rightarrow \begin{bmatrix} 1 & 1 & \vdots & 9 \\ 0 & -3 & \vdots & -15 \end{bmatrix}$ **Applying changes to row two[R2-2R1]**

$\Rightarrow \begin{bmatrix} 1 & 1 & \vdots & 9 \\ 0 & 1 & \vdots & 5 \end{bmatrix}$ **Applying changes to row two again [** $-\frac{1}{3}$**R2]**

$\Rightarrow \begin{bmatrix} -1 & 0 & \vdots & -4 \\ 0 & 1 & \vdots & 5 \end{bmatrix}$ **Applying changes to row one [R2-R1]**

$\Rightarrow \begin{bmatrix} 1 & 0 & \vdots & 4 \\ 0 & 1 & \vdots & 5 \end{bmatrix}$ **Applying changes to row one [-R1]**

$\Rightarrow \therefore x = 4 \quad and \; y = 5$

So augmented matrix just saves us carrying x's and y's through the manipulations.

TYPES OF ROW OPERATIONS
1.DIVISION OR MULTIPLY A ROW BY NUMBER OTHER THAN ZERO
2.SUBTRACT OR ADD A MUTIPLE OF ONE ROW FROM ANOTHER
3.EXCHANGE TWO ROWS

To keep track of all the row operations

- **We label each row of the augmented matrix starting with the top row as R1**
- **We note the actual row operation that has been performed**

So lets now do a 3x3 matrix example

So given

x+ y+ z =6……⊙

3x+4y+2z=17……..⊙

-x+2y+z=6…….⊙

Solution:

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \\ 6 \end{bmatrix} \; \boldsymbol{matrix\ notation}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 3 & 4 & 2 & \vdots & 17 \\ -1 & 2 & 1 & \vdots & 6 \end{bmatrix}$$ **augmented matrix**

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & -1 & \vdots & -1 \\ 0 & 3 & 2 & \vdots & 12 \end{bmatrix} \; \boldsymbol{Apply\ changes\ to\ row\ two}$$ **R2-3R1 then row three R1+R2**

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & -1 & \vdots & 17 \\ 0 & 0 & 5 & \vdots & 15 \end{bmatrix} \; \boldsymbol{Apply\ changes\ to\ row\ three}$$ **R3-3R2**

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & -1 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \; \boldsymbol{Apply\ changes\ to\ row\ three} \frac{1}{5} R3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$ **Apply changes to row two R2+R3**

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 & \vdots & -4 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$ **Apply changes to row one R2-R1**

$\Rightarrow \begin{bmatrix} -1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix}$ **Apply changes to row one again R1+R3**

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix}$ **Apply changes to row one again -R1**

∴ x= 1 & y=2 then z=3

Exercise 2

1.solve each of the following system of equations using the Gauss-Jordan row reduction technique

(a) -4x+3y=8

3x-2y=12

(b) -x-2y+3z = -4

y-z=6

2x+y-2z=2

[note you solved these in exercise 1 using $X=A^{-1}B$ **]**

2.use the Gauss-Jordan row reduction technique to solve the following

(a) x-y+ z=3

2x+3y-4z=9

-x+2y-z=0

(b) -x+2y+3z=10

3x-2y+z=-16

-2x-y-4z=5

(c) x +y +z=1

2x -y+4z=-2

-x+3y-3z=-7

(d) 2x-y=7

-x+2y=-5

Please attempt these problems and apply direct substitution after you have found the values of x, y, z to check your solution; remember practice makes perfect. By the time you stay longer with the problem the better you understand it. Now we will do the inverses of matrix

SOLUTION OF LINEAR EQUATIONS USING THE INVERSE OF THE MATRICES

One ways of finding the inverse is to use the Gauss-Jordan technique you have recently mastered.

NB
If B is the inverse of A then AB=I ,Let say A is (2X2) then B is (2X2) and I is I_2

i.e. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 0 & 1 \end{bmatrix}$
then since B is the inverse of A, we must find the values of b_{11} , b_{12} , b_{21}, b_{22} this is similar to finding x and y when we solved

Example

Find the inverse of the matrix A=$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

Solution: we know that AA^{-1}=I_2

Form $[A \;\vdots\; I_2]$

$\Rightarrow \begin{bmatrix} 1 & 1 & \vdots & 1 & 0 \\ 3 & 2 & & 0 & 1 \end{bmatrix}$ **R1 AND R2**

$\Rightarrow \begin{bmatrix} 1 & 1 & \vdots & 1 & 0 \\ 0 & -1 & & -3 & 1 \end{bmatrix}$ **APPLY CHANGES TO ROW TWO [R2-3R1]**

$\Rightarrow \begin{bmatrix} 1 & 0 & \vdots & -2 & 1 \\ 0 & 1 & & 3 & -1 \end{bmatrix}$ **APPLY CHANGES TO ROW ONE [R1+R2] THEN APPLY CHANGES TO ROW TWO R2×(-1)**

LHS Of augmented matrix has been reduced to I_2 so RHS of augmented matrix gives A^{-1},the inverse of A

A^{-1} =$\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$

Check solution by showing AA^{-1}=I_2 or $A^{-1}A$=I_2

AA^{-1} = $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$

=$\begin{bmatrix} (1\times -2+1\times 3) & (1\times 1+1\times -1) \\ (3\times -2+2\times 3) & (3\times 1+2\times \ -1) \end{bmatrix}$

=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$=$I_2$

EXAMPLE 2

2u+3v-2w=5

2u+v+w=11

3u+2v-3w=0

Solution:

AX=B where A=$\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$ **, X=**$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ **and B=**$\begin{bmatrix} 5 \\ 11 \\ 0 \end{bmatrix}$

If AX=B

$\therefore X = A^{-1}B$

$[A \;\vdots\; I_3]$ =$\begin{bmatrix} 2 & 3 & -2 & & 1 & 0 & 0 \\ 2 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 3 & 2 & -3 & & 0 & 0 & 1 \end{bmatrix}$ **R1 ,R2 and R3 respectively**

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 0 & 0\\ 0 & -2 & 3 & \vdots & -1 & 1 & 0\\ 0 & -\frac{5}{2} & 0 & \vdots & -\frac{3}{2} & 0 & 1\end{bmatrix}$$ **Apply changes to row one [R1/2], Apply changes to row two**

[R2-2R1] then Apply changes to row three [R3-3R1]

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 0 & 0\\ 0 & 1 & -\frac{3}{2} & \vdots & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{15}{4} & \vdots & -\frac{1}{4} & -\frac{5}{4} & 1\end{bmatrix}$$ **Apply changes to row two [R2/(-2)] then Apply changes to row three [R3+$\frac{5}{2}$R2]**

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{2} & 0 & \vdots & \frac{17}{30} & \frac{1}{3} & -\frac{4}{15}\\ 0 & 1 & 0 & \vdots & \frac{3}{5} & 0 & -\frac{2}{5}\\ 0 & 0 & 0 & \vdots & \frac{1}{15} & \frac{1}{3} & -\frac{4}{15}\end{bmatrix}$$ **Apply changes to row one [R1+R3] ,Apply changes to row two [R2+$\frac{3}{2}$ R3] then Apply changes to row three [R3×(- $\frac{4}{15}$)**

$$\Rightarrow\begin{bmatrix}1 & 0 & 0 & \vdots & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 0 & 1 & 0 & \vdots & \frac{3}{5} & 0 & -\frac{2}{5}\\ 0 & 0 & 0 & \vdots & \frac{1}{15} & \frac{1}{3} & -\frac{4}{15}\end{bmatrix}$$ **Apply change two row one [R1-$\frac{3}{2}$R2]**

Now the LHS of augmented matrix is I_3, so the RHS must be A^{-1} ,Assuming we've made no errors ! before proceeding to find X, let check that the A^{-1} is correct by showing AA^{-1}=I_3

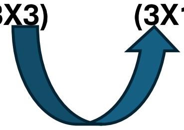
$$\therefore AA^{-1} = \begin{bmatrix}2 & 3 & -2\\ 2 & 1 & 1\\ 3 & 2 & -3\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{3}{5} & 0 & -\frac{2}{5}\\ \frac{1}{15} & \frac{1}{3} & -\frac{4}{15}\end{bmatrix}\begin{bmatrix}(-\frac{2}{3}+\frac{9}{5}-\frac{2}{15}) & (\frac{2}{3}+0-\frac{2}{3}) & (\frac{2}{3}-\frac{6}{5}+\frac{8}{15})\\ (-\frac{2}{3}+\frac{3}{5}+\frac{1}{15}) & (\frac{2}{3}+0+\frac{1}{3}) & (\frac{2}{3}-\frac{2}{5}-\frac{4}{15})\\ (1-\frac{4}{5}+\frac{4}{5}) & (1+0-1) & (1-\frac{4}{5}+\frac{4}{5})\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I_3$$

Thus we know A^{-1} *is correct.*

$$\therefore A^{-1}\mathbf{B}=\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & -\frac{2}{5} \\ \frac{1}{15} & \frac{1}{3} & -\frac{4}{15} \end{bmatrix}\begin{bmatrix} 5 \\ 11 \\ 0 \end{bmatrix}=\begin{bmatrix} (-\frac{5}{3}+\frac{11}{3}+0) \\ (3+0+0) \\ (\frac{1}{3}+\frac{11}{3}+0 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(3X3) (3X1) [Same inner dimensions therefore product defined]



∴ u=2 ; v=3 and w=4

which is like the solution we obtain earlier. In my research Inverses of matrices have a= wide application in mathematics (not just for solving simple linear systems) so I suppose you spend some time on this section before proceeding. Make sure you do all the exercises

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Exercise 3

1.Solve this set of simultaneous equations using the inverses of the matrix of coefficients

x +2y+3z=6

2x+4y+5z=9

3x+5y+6z =1

2. verify your solution please

3. For each problem in Exercise 2, Find the inverse of coefficients, A . check your inverse by showing

AA^{-1} **=I**

Determinants of a Square Matrix

Before I end this document I wanted to introduce you to a characteristic of a square matrix called the determinant . If a square matrix has a determinant which is zero ,it will not have an Inverse.so before doing the work to find the inverse of a matrix it is good idea to check if the determinant of a matrix is non zero, there are several methods for finding the determinant of a matrix. We will use the following method.

Example

If A is a (2X2) matrix

i.e. A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ now most importantly we have to follow the matrix rule whenever we dealing with determinants the rule state that $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ which means the plus and minus signs must alternate along the each row and column so

the determinant of A i.e. det(A)= $a_{11}a_{22} - a_{12}a_{21}$

More examples

So let get started if A=$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$,find the determinant of A, will A have an inverse ?

Solution:

Det(A)=(3× 1) −(-2×-1)

=3−2

=1

∴as det(A) is non zero (not equals to zero), A will have an inverse .so this matrix is invertible

Example 2[3X3]

i.e. A=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$**,NB A=**$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

SO det(A)= $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Now let try and apply

If A=$\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, find the determinant of A. will A have an inverse ?

Det (A)=$2\times \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

Now use method for (2X2) matrix to workout the determinants of the (2X2) matrices

Det(A)=2(1×-3-1×2)-3(2× −3 −1× 3)−2(2×2−1×3)

=−10+27−2

=15

$\therefore$ *as the det(A)is non zero , A will have an inverse*

Exercise 4

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1. Find the determinants to determine which of the following matrices have an inverse.

   (a)$\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$ (b)$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 4 & 6 & 3 \\ 2 & 7 & -1 & 5 \\ 4 & 2 & -3 & 1 \end{bmatrix}$ [to test if you really get the concept of determinants

Thank you!, I’m glad if you found this document helpful,

References

1.”linear Algebra and its Applications” by Gilbert Strang

2.”linear Algebra: A modern introduction” by David Poole